Vladimir Protasov
L'Aquila (Italy), MSU, HSE (Russia)

Optimization tools in the Lyapunov stability problem

Stability of linear systems

A linear dynamical system with continuous time:

$$\dot{x}(t) = A\,x(t), \quad t \in [0,+\infty), \quad x(0) = x_0$$

A system is stable if all trajectories tend to zero
(Hurwitz stability)

$$\mathrm{Re}\,\lambda_k < 0\,, \quad \lambda_k \in sp(A)$$

A linear dynamical system with discrete time:

$$x_{k+1} = A\,x_k\,, \quad k \in \mathbb{Z}_+ \quad x_0 \text{ is given}$$

A system is stable if all trajectories tend to zero
(Schur stability)

$$|\lambda_k| < 1\,, \quad \lambda_k \in sp(A)$$

Positive linear systems.

How to find the closest stable/unstable system?

Applications

Mathematical economics (Leontief input-output model)

Population dynamics in mathematical biology

Epidemiological dynamics

Fluid network control

Blanchini, Colaneri, Valker "*Switched positive linear systems*", 2015

Krause, "*Swarm dynamics and positive dynamical systems*", 2013

Let A be a d x d matrix, $\rho(A)$ be its spectral radius.

If $\lambda_1, \ldots, \lambda_d$ are eigenvalues of A and $|\lambda_1| \geq \ldots \geq |\lambda_d|$, then $\rho(A) = |\lambda_1|$.

Theorem 1. (Perron (1906), Frobenius (1913)). If $A \geq 0$, then the spectral radius is attained at a real positive eigenvalue $\rho(A) = \lambda_1 \geq 0$.

There is an eigenvector $v \geq 0$ such that $Av = \lambda_1 v$.

If $A > 0$, then the largest by modulo eigenvalue is unique and simple, and so is the corresponding eigenvector.

$$\rho(A) = \lambda_1$$

We call $\lambda_{max} = \lambda_1 = \rho(A)$ the leading eigenvalue and v the leading eigenvector.

A linear dynamical system with discrete time:

$$x_{k+1} = A\,x_k, \quad k \in \mathbb{Z}_+ \quad x_0 \text{ is given}$$

A system is Schur stable if $\lambda_1 < 1$

Problems

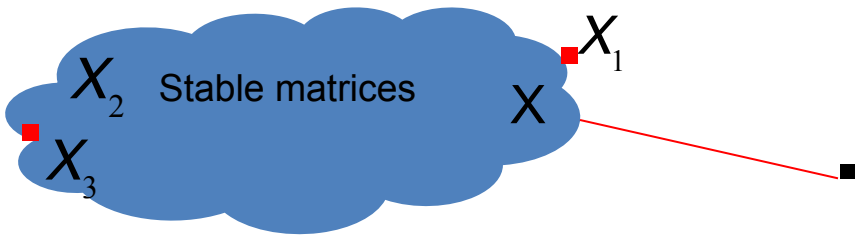
How to find the closest stable matrix to A ?
How far is A to the set of stable matrices ?

How to find the closest unstable matrix to A ?
How far is A to the set of unstable matrices ?

$$\|X - A\| \to \min$$
$$\rho(X) = 1$$

This is equivalent to the optimizing of the spectral radius of a matrix over a matrix ball

$$\|X - A\| \to \min$$
$$\rho(X) = 1$$



$X \in absmin$

$X, X_1, X_2, X_3 \in locmin$

A lot depend on the norm in $\mathbb{R}^d$

We consider the Frobenius norm

$$(A, B) = tr\, A^T B\ ; \qquad \|A\|^2 = (A, A) = \sum_{i,j=1}^{d} a_{ij}^{\ 2}$$

R. Byers (1988)

F.X. Orbandexivry, Y. Nesterov, and P. Van Dooren, (2013)

C. Mehl, V. Mehrmann, and P. Sharma (2016)

N. Gillis and P. Sharma (2017)

J. Anderson (2017)

The problem is nasty. No efficient algorithms are available.

There are examples of d x d matrices that have at least 2^d locally closest stable matrices (Guglielmi, V.P., 2018).

Optimizing the spectral radius of a matrix

The general problem:

$$\begin{cases} \rho(A) \rightarrow \max/\min \\ A \in M \end{cases}$$

M is a set of marices.

The nonnegative case:

$$\begin{cases} \lambda_{\max}(A) \rightarrow \max/\min \\ A \in M \end{cases}$$

M is a set of nonnegative marices.

These problems are both notoriously hard
(even if the set M is convex).

Reasons:

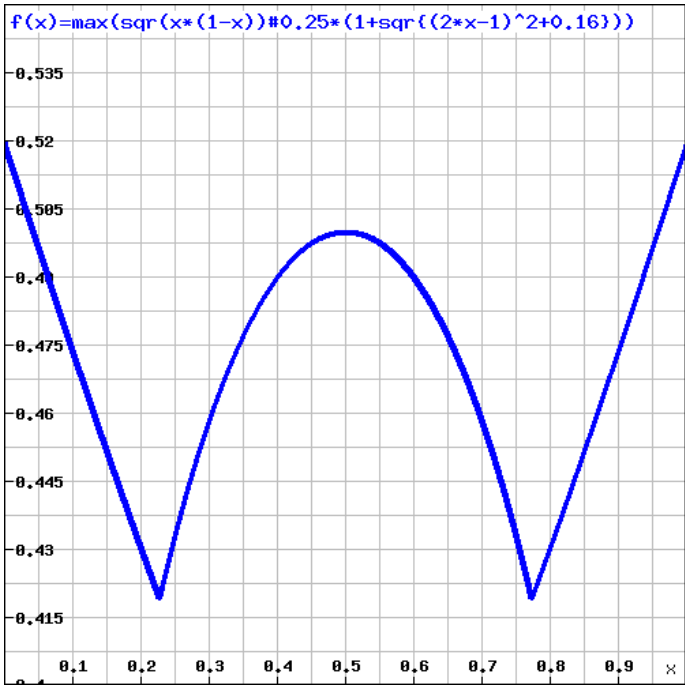
- The spectral radius is neither convex nor concave in matrices
- The spectral radius is non-Lipschitz, if the leading eigenvalue is multiple.

Example 1. For the set $M = [A_1, A_2] = co\{A_1, A_2\}$,

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.1 \\ 0 & 0 & 0.1 & 0 \end{pmatrix} \quad ; \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0.1 & 0.5 \end{pmatrix}$$

We have $A = (1-x)A_1 + x\,A_2, \quad x \in [0,1]$. ``Matrix segment"

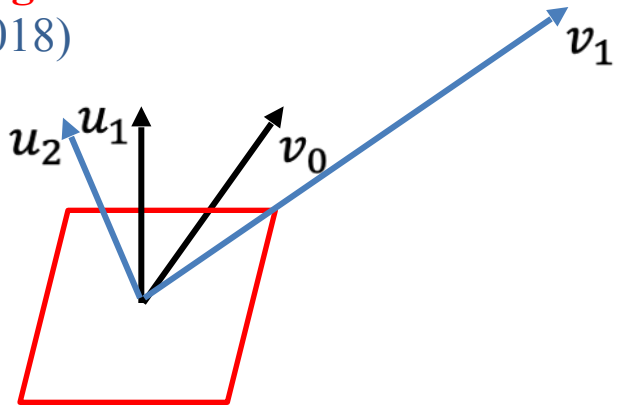
$\lambda_{\max}(A) =$



The algorithm of alternating relaxations

(N.Guglielmi, V.P., 2018)

Let A be a non-stable matrix, $\rho(A) > 1$.



1)Take a matrix X_0, $\rho(X_0) = 1$, v_0 is the right leading eigenvector: $X_0 v_0 = v_0$

X_1 is the solution of the problem $\|X - A\| \to \min$

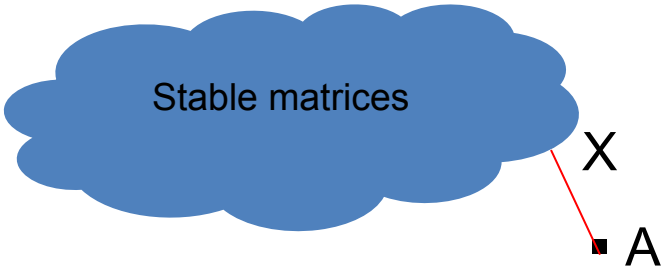
$X v_0 = v_0$

2)Take u_1 be the left leading eigenvector: $u_1 X_1 = u_1$

X_2 is the solution of the problem $\|X - A\| \to \min$

$u_1 X = u_1$

$X \in absmin$



Then we alternate left and right leading eigenvectors. The distance to A decreases each step.

Theorem. If the algorithm converges to a positive matrix X, then X is a global minimum. In this case the convergence is linear and the rate can be estimated from above.

The curse of sparsity

If the limit matrix X have some zero components, then

$$X = \begin{pmatrix} X_1 & * \\ 0 & X_2 \end{pmatrix}$$

1) If X is irreducible, then X is a local minimum;

2) If X is reducible, then the we obtain several problems of smaller dimensions. We solve them separately

Theorem. If the algorithm always converges to a local minimum with a linear rate.

Example.

$$A = \begin{pmatrix} 0.7 & 0.2 & 0.1 & 0.5 & 1.0 \\ 0.3 & 0.6 & 0.2 & 0.8 & 0.3 \\ 0.5 & 0.7 & 0.9 & 1.0 & 0.5 \\ 0.1 & 0.1 & 0.3 & 0.8 & 0.3 \\ 0.8 & 0.2 & 0.9 & 0.3 & 0.2 \end{pmatrix}, \qquad \text{with} \quad \rho(A) = 2.4031.$$

$$X_2 = \begin{pmatrix} 0 & 0.4204 & 0.1759 & 0.6770 & 0.3 \\ 0.7343 & 0.3796 & 0.1797 & 0 & 0.5 \\ 0.0274 & 0 & 0.5791 & 0.0069 & 0.8 \\ 0.1334 & 0.0580 & 0.6719 & 0.6403 & 1 \\ 0 & 0 & 0 & 0 & 0.8 \end{pmatrix}$$

Trying another norm ?

Y.Nesterov, V.P (2018)

We consider the closest stable nonnegative matrix in two polyhedral forms in $\mathbb{R}^d$:

The $L_\infty - \text{norm}$: $\|A\|_\infty = \max\limits_{i=1,\dots,d} \sum\limits_{j=1}^{d} |a_{ij}|$

The $L_1 - \text{norm}$: $\|A\|_1 = \max\limits_{j=1,\dots,d} \sum\limits_{i=1}^{d} |a_{ij}|$

The problem becomes:

Find the minimal $\tau > 0$ with the following property:

The matrix can be made stable/unstable by chanding elements of each row by at most τ in the sum

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

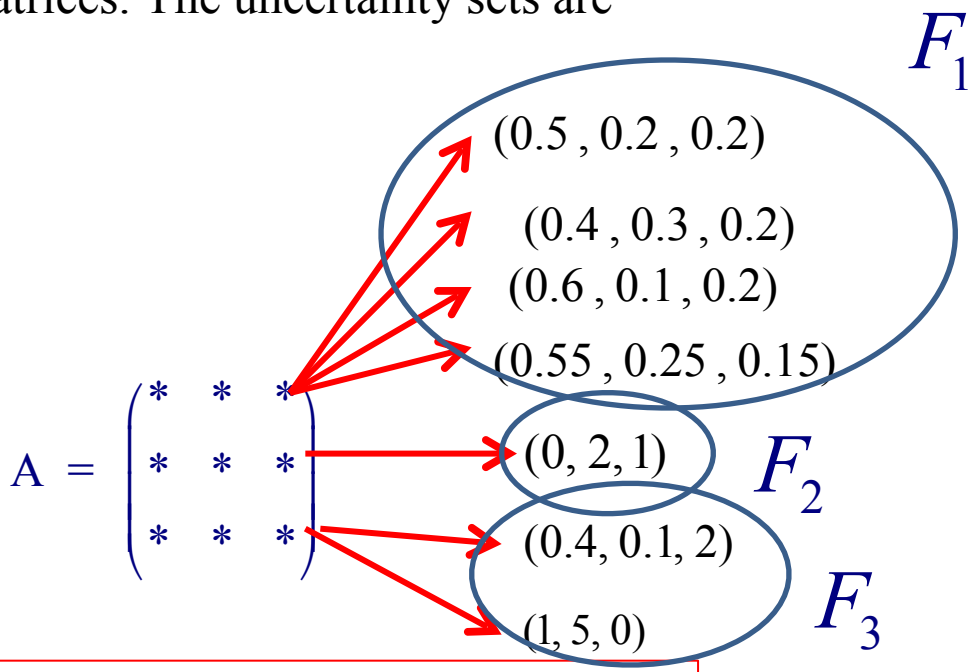
Note that rows of the matrix can be changed independently of each other.
Every row runs over an $L_1 - \text{ball}$ independently of others.

The unit ball of matrices in the L_∞ norm is a row uncertainty set

The row uncertainty sets of matrices

Definition 1. A family of matrices is called a *product family,* if the rows of matrices are chosen independently from given sets *(uncertainty sets)* F_i, $i = 1, \ldots, d$.

Example 2. A family of 3x3-matrices. The uncertainty sets are



We obtain the family M of 4x1x2 = 8 matrices

The exhaustion is hard!

If we have $d = 50$ and just TWO lines in each uncertainty set, then the total number of matrices is $2^{50} > 10^{15}$.

Moreover, the set of rows may be polyhedral
(a subset of R^d defined by a system of linear inequalities).

Product families with row uncertainties

V.Kozyakin (2004)
V.Blondel, Y.Nesterov (2009)
Y.Nesterov, V.P. (2013)

Applications:

- Leontief model (mathematical economics)
- Combinatorics
- Spectral graph theory
- Asyncronouos systems

Optimizing the spectral radius over product families

Studied in: Y.Nesterov, V.P. (2013), V.P. (2015), Cvetkovic (2019), M.Akian, S.Gaubert, J.Grand-Clément, and J.Guillaud (2017)

The spectral simplex method

Definition 2. A *one-line correction,* of a matrix is a replacement of one of its lines.

Example 3. A correction of the first line. We replace the row a_1 by some row a'_1.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \longrightarrow A' = \begin{pmatrix} a'_{11} & a'_{12} & a'_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$a'_1 = (a'_{11}, a'_{12}, a'_{13})$$

Theorem 2. Let M be a product family of strictly positive matrices, $F_1,\ldots,F_d$ be uncertainty sets. For every A $\in M$ with the leading eigenvalue λ and eigenvector v, we have

a) If there is $a'_i \in F_i$ such that $(v,\ a'_i) \ >\ (v,\ a_i)$, then after the one-line correction we have

$$\lambda_{\max}(A') \ > \ \lambda_{\max}(A)$$

b) If the matrix A is maximal in each row with respect to v, i.e., $(v,\ a_i) \ = \ \max\limits_{a'_i \in F_i} (v,\ a'_i), \qquad i = 1,\ldots,\ d,$ then

$$\lambda_{\max}(A) \ = \ \max_{A' \in M} \lambda_{\max}(A')$$

The spectral simplex method

Initialization. Take an arbitrary matrix $A_1 \in M$.

Main loop. We have a matrix A_k and its leading eigenvector $v_k > 0$.

For every $i = 1, \ldots, d$ do:

Step i. Find $a'_i = \arg\max_{b_i \in F_i} (v, b_i)$.

If $a'_i = a_i$, then set $A_{k+1} = A_k$ and go to the step $i+1$.

Otherwise, we have $(v, a'_i) > (v, a_i)$.

Make the one-line correction in the ith line.

Theorem 3 implies that $\rho(A'_k) > \rho(A_k)$.

Put $A_{k+1} = A'_k$. We have $\rho(A_{k+1}) > \rho(A_k)$.

Compute the leading eigenvector $v_{k+1} > 0$ of A_{k+1}.

Go to step $i = 1$.

If the dth step is over, then END.

Theorem 3. For strictly positive matrices, the spectral simplex method is well-defined, does not cycle, and finds the solution within finite time.

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In many problems, the matrices are sparse. In this case we are in trouble.

- The leading eigenvector v of a matrix A may not be unique.
- The spectral radius is not strictly increasing with iteration, but just non-decreasing

The algorithm may cycle.

For sparse matrices, the algorithm cycles very often.

Theorem 4. Assume a nonnegative matrix A has a simple leading eigenvector $v \geq 0, \quad \| v \| = 1.$
Then after an arbitrary one-line correction such that $(v, a'_i) > (v, a_i)$,
the matrix A' possesses the same property.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \longrightarrow A' = \begin{pmatrix} a'_{11} & a'_{12} & a'_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$a'_1 = (a'_{11}, a'_{12}, a'_{13})$ such that $(v, a'_1) > (v, a_1).$

Theorem 5. If the initial matrix A_1 of the spectral simplex method has a simple leading eigenvector, then all matrices in all iterations possess the same property, and the algorithm does not cycle.

How to choose A_1 to possess a unique leading eigenvector ?

For instance to take the kth row of A_1 to be the arithmetic mean of all rows from the uncertainty set F_k, for each $k = 1, .., d.$

The numerical efficiency of the spectral simplex method

Table 1 The number of iterations for maximizing the spectral radius of positive $d \times d$ matrices

The sets $\mathcal{F}_i$ are finite, each has n elements

d/n	2	5	10	50	100
5	3	6	8	10	13
10	7	12	14	18	23
50	29	48	58	92	109
100	56	99	131	197	213
500	274	542	701	884	1034

For $d = 100,\ n = 2,$ we have the 100-dimensional Boolean cube.

The number of vertices is 2^{100}. However, the algorithm performs only 56 one-line corrections.

t = 12 s.

For $d = 10,\ n = 100,$ the set M contains $100^{10} = 10^{20}$ matrices. The algorithm performs 23 iterations.

t = 0.3 s.

For $d = 100,\ n = 100,$ the set M contains 10^{200} matrices. The algorithm performs 213 iterations.

t = 40 s.

Theorem 6. For a product family M of strictly positive matrices, there are constants $C > 0,\ q \in (0,1)$, such that

$$\left|\rho(A_N) - \rho(A_*)\right| \leq C\, q^N,$$

where A_* is the optimal matrix, A_N is the matrix obtained in the Nth iteration of the spectral simpex method.

What happens if we optimize not one row but all rows simultaneously?

For small dimensions (d=2,3) we got worse results (3-4 iterations).
We have arguments for that.

M.Akian, S.Gaubert, J.Grand-Clément, and J.Guillaud, The operator approach to entropy games (2017)

The greedy algorithm converges with a fantastic rate!

$d \setminus N$	50	100	250
25	5.5	6.2	6.3
100	4.2	4.5	4.6
500	4.1	4.3	4.3
2000	4.1	4.3	4.1

Bless of dimensionality ?

Theorem (Cvetkovic, V.P., 2018) The greedy algorithm has a quadratic rate of convergence, provided all the uncertainty sets are strictly positive.

$$\left\| X_{k+1} - X \right\| \leq B \left\| X_k - X \right\|^2$$

$B \leq \frac{C R}{2 r \rho(X)}$, where R and r are maximal and minimal curvature radii of two-dimensional cross-sections of ∂F_i

The cycling phenomenon

Example

$$\mathcal{F}_1 = \left\{ \begin{array}{ccc} (1, & 1, & 1) \\ (0, & 5, & 10) \\ (0, & 10, & 5) \\ (12, & 0, & 0) \end{array} \right\} \qquad \mathcal{F}_2 = \left\{ \begin{array}{ccc} (1, & 1, & 1) \\ (0, & 10, & 0) \end{array} \right\}; \qquad \mathcal{F}_3 = \left\{ \begin{array}{ccc} (1, & 1, & 3) \\ (0, & 0, & 10) \end{array} \right\}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix} \xrightarrow{\boldsymbol{v}_1=(1,1,2)^T} \begin{pmatrix} 0 & 5 & 10 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix} \begin{array}{c} \xrightarrow{\boldsymbol{v}_2=(2,2,1)^T} \\ \\ \xleftarrow{\boldsymbol{v}_3=(2,1,2)^T} \end{array} \begin{pmatrix} 0 & 10 & 5 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

Anti-cycling modification. The selective greedy method

Def. The selected leading eigenvector of a matric X is the limit of eigenvectors of matrices $X + I + \varepsilon E$ as $\varepsilon \to +0$, where I is the identity matrix and E is the matrix of ones.

Theorem 1. The selective greedy method does not cycle.

Theorem 2. The selected leading eigenvector is the limit of the power method $x_{k+1} = Ax_k$ with the initial vector of ones $x = e = (1,...,1)^T$.

To find the closest stable matrix, one fix $\tau > 0$, solves the problem

$$\rho(\mathrm{X}) \to \min$$
$$\left\| X - A \right\|_\infty = \tau$$

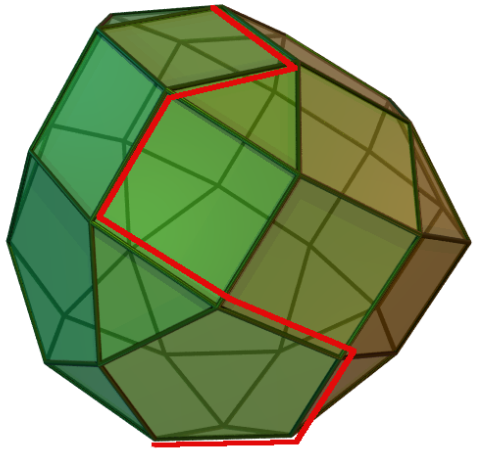
and the find the smallest τ by bisection.

The classical simplex method (for linear programming, G.Dantzig, 1947).

LP problem: $\begin{cases} (c,x) \rightarrow \max \\ (a_i,x) \le b_i, \quad i=1,\dots,N \end{cases}$

Step-by-step increasing of the objective function (c,x) going along the edges of the polyhedron

$$G = \{ (a_i,x) \le b_i, \quad i=1, \dots, N\}.$$



In practice, converges extremely fast.
G.Dantzig believed that the number of steps is linear in N and d.

1972. V.Klee and G.Minty constructed an example with 2^N iterations.

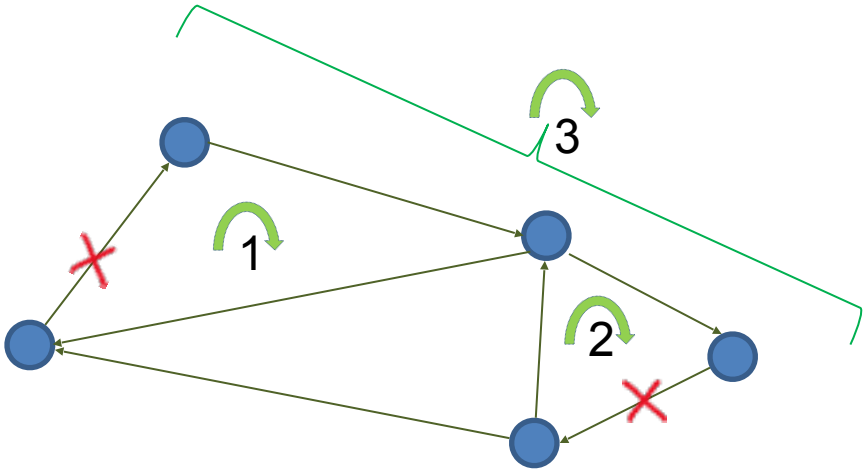
In average, the number of iteration is indeed linear in N and d (S.Smale, 1983).

What is the theoretical complexity of the **(greedy) Spectral simplex method ?**

The Maximal Acyclic Subgraph problem (MAS)

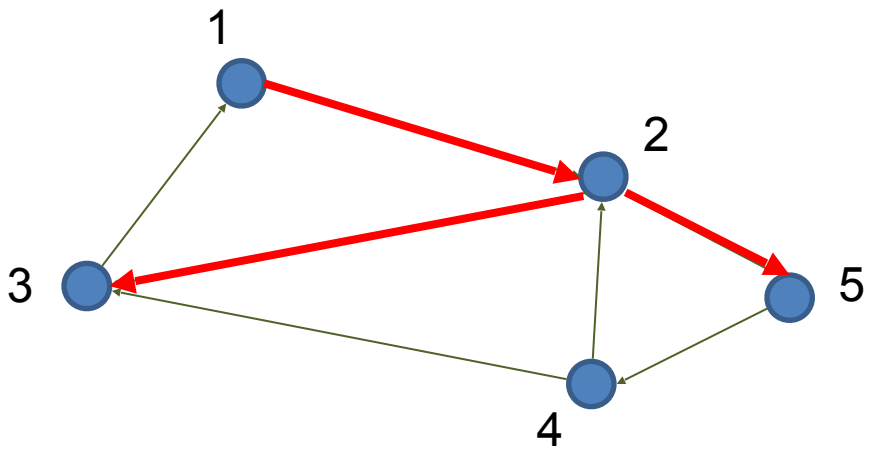
Let $G = (V, E)$ be a given directed graph.

Find its acyclic subgraph $G' = (V, E')$ for which $|E'| \to \max$

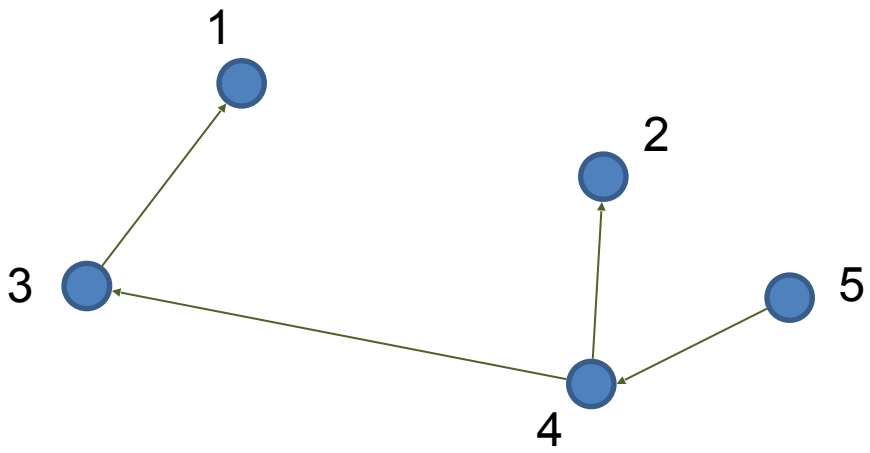


Answer: $\max |E'| = 5$

The simplest method: make some ordering of vertices, then take those edges E' directed in the increasing order (or decreasing). Then $G' = (V, E')$ is acyclic.



We have $|E'| = 3$



For the decreasing edges |E'| = 4

At least one of these two sets of edges contains $\geq \frac{1}{2} |E|$ edges.

Therefore, this simple method gives an approximate answer with factor $\geq \frac{1}{2}$

This is still the best approximation obtained by a polynomial algorithm.

No polynomial algorithm is known with approximation factor $\frac{1}{2} + \varepsilon$

Finding max |E'| is NP complete

The MAS problem is in the list of 21 NP-complete problems by R.Karp (1973)

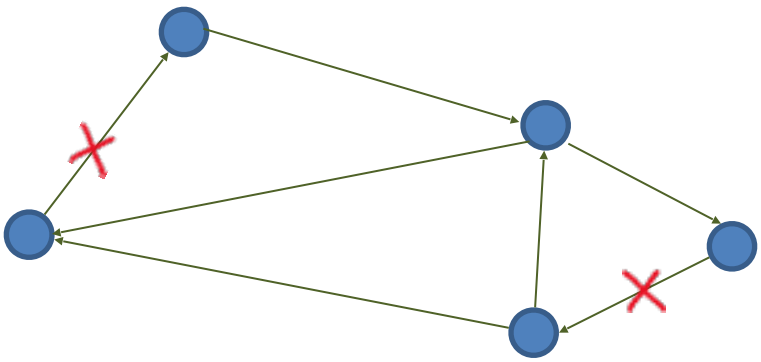
Finding an acycling subgraph

with $\geq \left(\frac{65}{66} + \varepsilon\right) \max |E'|$ is NP hard

There are algorithms that give with approximation factor $\frac{1}{2} + \varepsilon$

for the vast majority of graphs.

Observation: A graph is acyclic $\Leftrightarrow$ $\rho(A) = 0$



$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

X is the MAS, if and only if X is the sulution of the problem

$$\|A - X\|_2 \to \min$$

$$\rho(X) = 0$$

This problem is closely related to the stabilising problem for positive dynamical systems:

$$\|A - X\|_2 \to \min$$

$$\rho(X) = 1$$

Approximate solution of MAS by the greedy method

d	50	250	500	1000	1500
time	0.36s	8.1s	66.42s	622.43s	2860.79s
# steps	17	34.6	38.5	44.7	50.3
γ	0.644	0.621	0.615	0.616	0.616

Table 1. Solving the max-MAS and approximating MAS for graphs with sparsity $9-51\%$

d	50	250	500	1000	1500
time	0.35s	6.56s	61.06s	605.73s	2614.02s
# steps	18.9	32.1	41.8	43.1	43.7
γ	0.6	0.592	0.592	0.593	0.592

Table 2. Solving the max-MAS and approximating MAS for graphs with sparsity $26-95\%$

Discrete switching systems

$$x_{k+1} = A(k)x_{k+1}\ ,$$
$$A(k) \in \{A_1, \dots, A_m\}\ ,\ k \geq 1,$$
$$x_0 \in \mathbb{R}^d$$

A system is stable if every trajectory $\{x_k\}_{k \in \mathbb{N}}$ tends to zero as $k \to \infty$

(The Schur stability)

Theorem (N.Barabanov, 1988) A discrete system is stable if and only if its joint spectral radius is smaller than one.

The Joint spectral radius (JSR)

$A_1, \cdots A_m$ are linear operators in $\mathbb{R}^d$

$$\hat{\rho}(A_1 \cdots, A_m) = \lim_{k\to\infty} \max_{d_1,\ldots,d_k \in \{1,\ldots,m\}} \left\| A_{d_1} \cdots A_{d_k} \right\|^{1/k}$$

J.C.Rota, G.Strang (1960) -- Normed algebras

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J.C.Rota, G.Strang (1960) -- Normed algebras

N.Barabanov, V.Kozyakin, E.Pyatnitsky, V.Opoytsev, L.Gurvits, …(1988)

Linear switching systems

C.Micchelli, H.Prautzch, W.Dahmen, A.Levin, N.Dyn, P.Oswald,…… (1989)

Subdivision algorithms

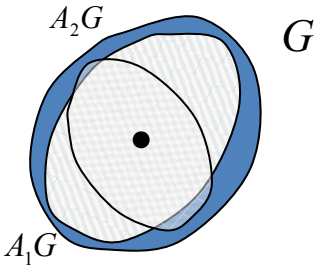
I.Daubechies, J.Lagarias , C.Heil, D.Strang, … (1991)

Wavelets

The Joint spectral radius (JSR)

$A_1, \boxed{?}, A_m$ are linear operators in $\mathbf{R}^d$

$$\hat{\rho}(A_1, \boxed{?}, A_m) = \lim_{k\to\infty} \max_{d_1,\dots,d_k\in\{1,\dots,m\}} \left\| A_{d_1} \quad \boxed{?} \quad A_{d_k} \right\|^{1/k}$$



The geometric sense:

$\hat{\rho} < 1 \quad \Leftrightarrow \quad$ there exists a norm $\|\bullet\|$ in $\mathbf{R}^d$

such that $\|A_i\| < 1$ for all $i = 1, \dots, m$

JSR is the measure of simultaneous contractibility

Taking the unit ball in that norm:

The JSR is smaller than 1 if and only if there is convex body G such that $A_k G \subset int\, G$, k = 1, .., m

Example. If all the matrices $A_1,\dots,A_m$ are symmetric, then

one can take G a Euclidean ball $\Rightarrow \quad \hat{\rho} = \max\{\rho(A_1),\dots,\rho(A_m)\}$

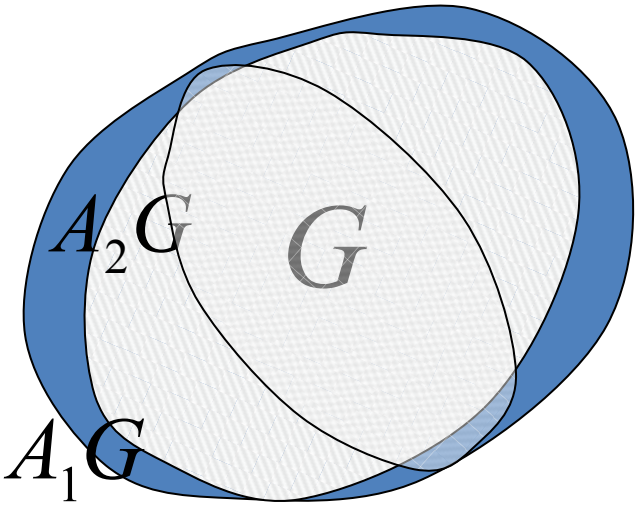
Other applications of the Joint Spectral Radius

- Probability
- Combinatorics
- Number theory
- Mathematical economics
- Discrete math

How to compute or estimate ?

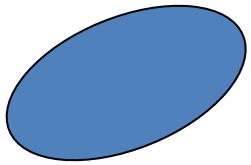
Blondel, Tsitsiklis (1997-2000).

- The problem of JSR computing for nonnegative rational matrices in NP-hard
- The problem, whether JSR is less than 1 (for rational matrices) is algorithmically undecidable whenever $d > 46$.
- There is no polynomial-time algorithm, with respect to both the dimension d and the accuracy


A_2G
G
A_1G

- ``The best ellipsoidal norm"

Once we are not able to find an extremal norm,
we find the best possible one in a class of norms.



Ando, Shih (1998),

Blondel, Nesterov, Theys (2004), the semidefinete programming framework

P., Jungers, Blondel (2010), the conic programming framework

We take all possible norms of the form $\|u\| = \sqrt{(Xu,u)}$, where X is a positive definite matrix.

$r_k \rightarrow \min$

subject to:

$\exists\ X \succeq 0$

$A^T X A - r_k X \preceq 0 \quad , \quad A = A_{d_1} \oplus \cdots \oplus A_{d_k}$

Theorem. For any k we have $d^{-1/k}\, r_k^{1/(2k)} \le \hat{\rho} \le r_k^{1/(2k)}$

- ``Tensor products of matrices" (P. (1997), Zhow (1998), Blondel, Nesterov (2005))

  Approximates the extremal norm by even polynomials

  Fast, but very rough

- ``Sum of squares algorithm" (Parrilo, Jadbabaie (2008))

  Approximates the extremal norm by some of squares polynomials.

  More or less the same complexity as the previous method.

Sometimes easier to prove more

George Polya «Mathematics and Plausible Reasoning» (1954)

When trying to *prove* something, often a good strategy is to try to *prove more*.

When trying to compute something approximately, often a good strategy is to…

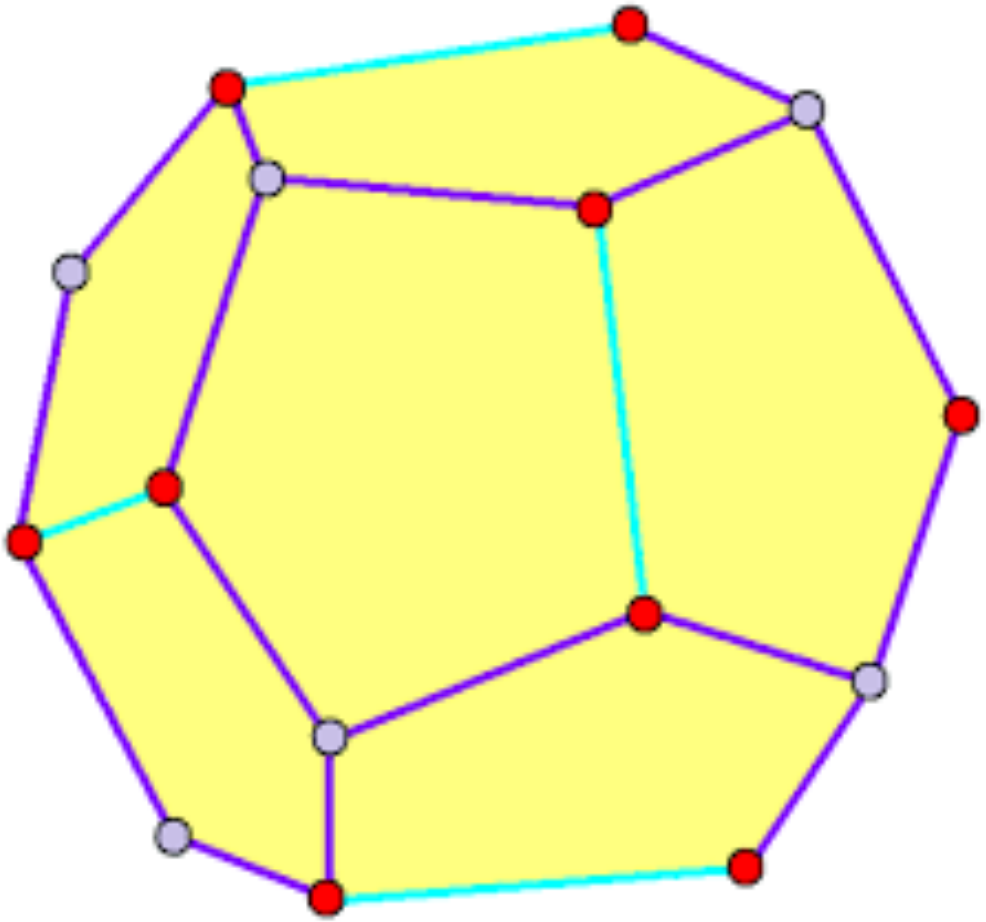
find it precisely.

N.Guglielmi, V.Protasov (2013)

The invariant polytope algorithm.

Normalize all operators so that $\rho(A_1, \dots, A_m) = 1$

and step-by-step construct a polytope G such that $A_k G \subset G$



Theorem. (N.Guglielmi, V.P., 2016)

The invariant polytope algorithm terminates within finite time if and only if the family has a finite number of dominant products

Invariant Lyapunov function (norm)

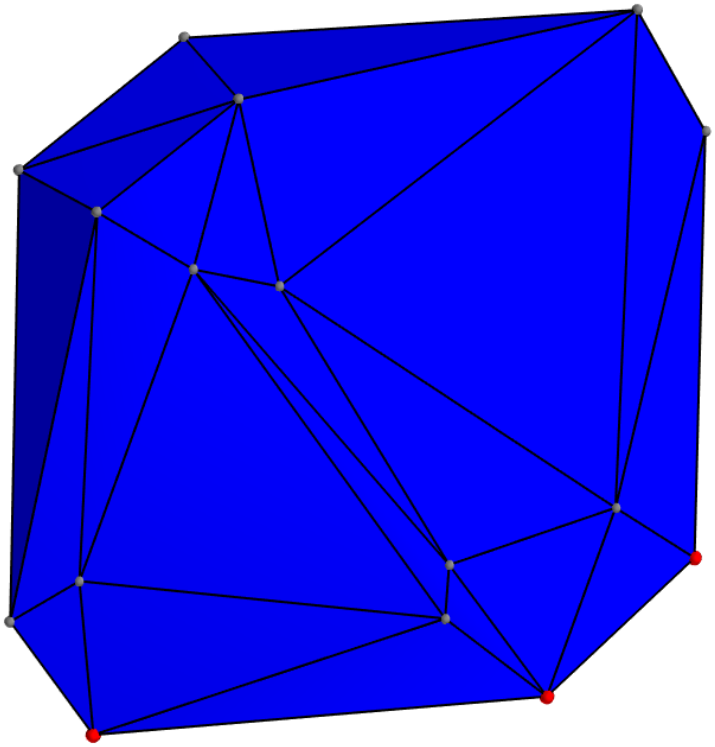
$$max_{k=1,..,m}\, f(A_k x) \;=\; \lambda\, f(x), \qquad x \in \mathbb{R}^d$$

$$\text{where} \quad \lambda \;=\; \rho(A_1, \ldots., A_m)$$

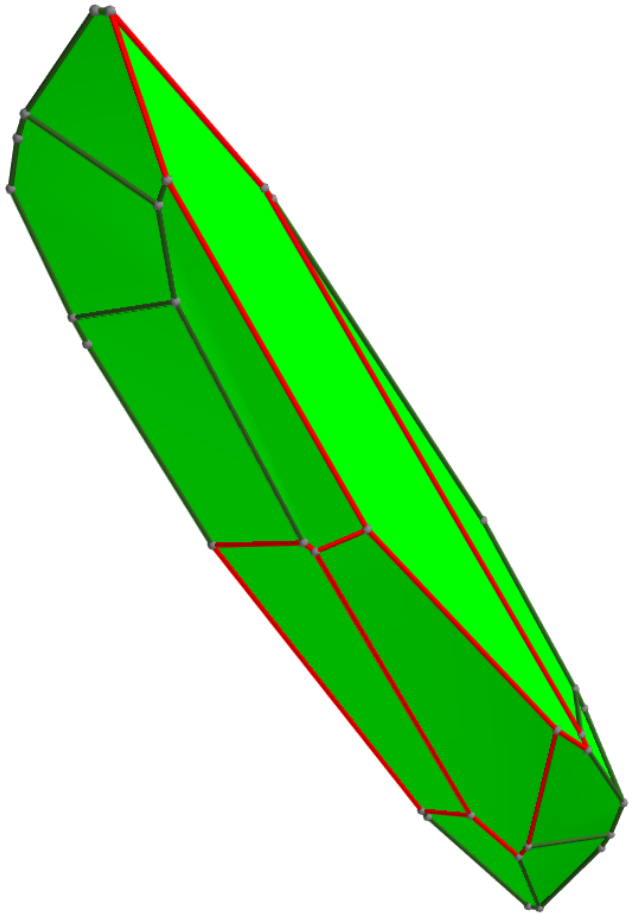
Theorem. F.Wirth (2008), N.Guglielmi, M.Zennaro (2019)

If P is an invariant polytope for the family of transposed matrices, then the polar P^* is the unit ball of the invariant Lyapunov norm $f(x)$.

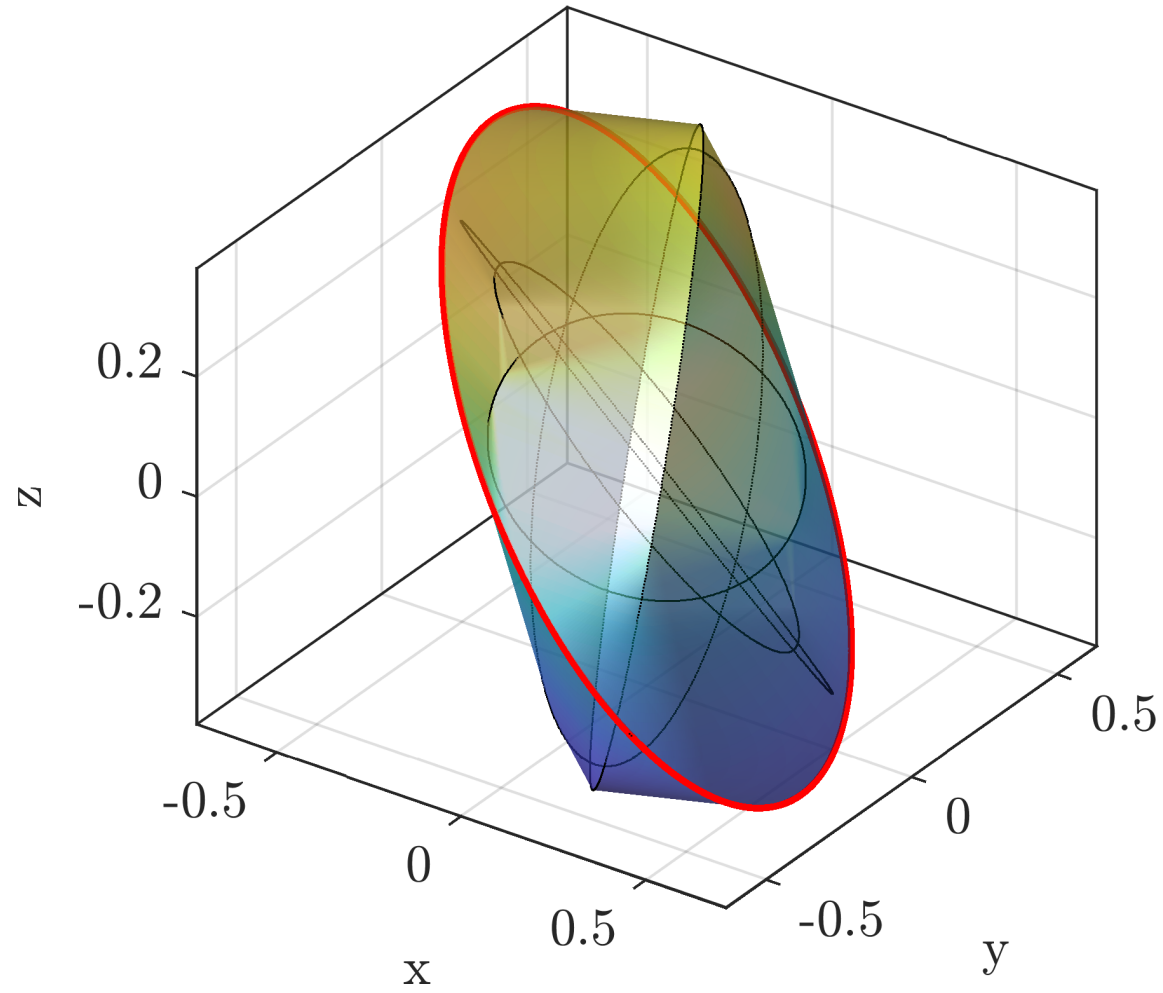
Invariant polytope **P**



The unit ball **P*** of the invariant Lyapunov norm



The complex case. The invariant elliptic polytope **P**



Thank you!