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Using the Mirror Descent Method for Online Optimization of Controlled Uncertain Dynamic Systems with Sliding Mode (Continuous Time)

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1 Introduction

Mirror Descent Method (MDM) (Nemirovsky and Yudin, 1979/1983) [1], cf. (Nesterov, 2007) [2], (Beck and Teboulle, 2003) [3], (Juditsky etal, 2005) [4] represents a non-trivial generalization of the standard gradient method to a problem

$$F(x) \;\rightarrow \min_{x \in X}\,, \tag{1}$$

$F : X \rightarrow \mathbb{R}$ be a convex function on convex compact $X \in \mathbb{R}^n$

We assume the 1st order oracle: $\forall\, t$, a *subgradient* $\nabla F(x_t)$ at current point $x_t \in X$ holds, that is

$$\langle \nabla F(x_t), x - x_t \rangle \leq F(x) - F(x_t), \quad \forall x \in X. \tag{2}$$

In continuous time $t \geq 0$, Inertial MDM (IMDM) (N.2018) [5], cf. (Nesterov and Shikhman, 2015) [6], contains the dual variable

$$\zeta_t = -\int_0^t \nabla F(x_\tau) \mathrm{d}\tau \, , \quad \zeta_0 = 0,$$

where primal variable x_t is defined by the ordinal differential equation (ODE)

$$\mu_t \dot{x}_t + x_t = \nabla W(\zeta_t), \quad t \geq 0.$$

Here the "mirror" mapping

$$\nabla W : \mathbb{R}^n \to X$$

is assumed to be smooth enough;
coefficient $\mu_t \geq 0$ has a sense of "mass".

Recall the inverse Fenchel-Legendre transformation

$$W_*(x) = \operatorname*{argmax}_{\zeta \in \mathbb{R}^n} \left\{ x^T \zeta - W(\zeta) \right\}, \quad x \in X.$$

Denote the optimal point $x^* \in \mathrm{Argmin}_{x \in X} F(x)$.

If we put $\mu_t = t$, then ODE

$$t\dot{x}_t + x_t = \nabla W(\zeta_t), \quad t \geq 0,$$

leads to the integral equation

$$x_t = \frac{1}{t} \int_0^t \nabla W(\zeta_\tau) \mathrm{d}\tau, \quad \mathrm{t} \geq 0,$$

with

$$\zeta_t = -\int_0^t \nabla F(x_\tau) \mathrm{d}\tau\,,$$

or, in the differential form

$$
\begin{aligned}
\dot{\zeta}_t &= -\nabla F(x_t), \quad \zeta_0 = 0, &(3)\\
t\dot{x}_t + x_t &= \nabla W(\zeta_t), \quad t \geq 0. &(4)
\end{aligned}
$$

Proposition 1 *For IMDM* (3)–(4)*, the inequality holds*

$$
F(x_t) - \min_{x \in X} F(x) \leq \frac{W_*(x^*)}{t}, \quad \forall t > 0.
$$

Suggesssion 1 *The IMDM* (3)*,* (4) *represents a closed system, and and allows us to consider it as a closed system with a static control plant.*

Further, we use this suggestion for the optimization problem of a dynamic control plant considered below.

Briefly recall that the Sliding Mode Control (SMC) is a variable structure control technique that employs intermittent control to slide along the boundaries of the control structures (in phase space), see (Utkin, 1992) [9]. Typical nonlinear control system (plant) is described by

$$\dot{\mathsf{x}}(t) = f(\mathsf{x}, t) + B(\mathsf{x}, t)u(t) \tag{5}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are state and control vectors, $m \leq$n. The functions $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $B : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times m}$ are assumed to be continuous and sufficiently smooth, to garantee the existence and uniquenness of the solution to (5). Below we consider a particular case of (5) with dimensions $n = 2m$.

2 Online optimization problem for an undefined dynamic plant

Consider a *dynamic controlled plant*

$$\ddot{x}_t = f\left(x_t, \dot{x}_t, t\right) + u_t, \quad t \geq 0, \tag{6}$$

where $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^n$ are state and control variables, and $f : \mathbb{R}^{2n} \times \mathbb{R}_+ \to \mathbb{R}^n$ represents an unknown dynamic function. The trajectories $\{x_\tau, \dot{x}_\tau\}_{\tau \in [0,t)}$ are assumed to be observable.

Also, consider the loss function $F(x_t)$, which characterizes the quality of control in (6) at the current time t for a given non-anticipatory control strategy $\mathcal{U}_t = u_t(\{x_\tau, \dot{x}_\tau\}_{\tau \in [0,t)})$, $t > 0$.

The control goal is to provide such a trajectory $\{x_t\}_{t>0}$ for which

$$F(x_t) \to \min_{x \in X} F(x) \quad \text{as} \;\; t \to \infty,$$

at convergence rate $O(1/t)$.

Basic assumptions:

A0. Solution $\{x(t)\}_{t\geq 0}$ to the equation (6) is assumed to exist and unique.

A1. Vector phase variable $(x_t, \dot{x}_t)$ is observable at each $t \geq 0$.

A2. The dynamics function f is unknown, but satisfies inequality

$$\|f(x,y,t)\|_2 \le c_0 + c_1 \|x\|_2 + c_2 \|y\|_2, \quad \forall (x,y) \in \mathbb{R}^{2n}, \ t \ge 0. \tag{7}$$

Constants $c_0 > 0$, $c_1 \ge 0$ and $c_2 \ge 0$ are known.

A3. Continuous loss function $F(x)$ is convex on the given bounded convex body X; therefore, $\forall\, x \in X$, a subgradient $\nabla F(x)$ exists (Rockafellar, 1970) [7].

Denote $x^* \in \operatorname*{Argmin}_{x \in X} F(x)$, $\quad F^* := \min_{x \in X} F(x)$.

In order to enable the purpose of the control plant, we use two ideas of IMDM (N.2018) [5]:

I1. We treat the observations of the subgradients $\nabla F(x_t)$ as vectors from the dual space $E_* = \mathbb{R}^n$ along with the original space of states $E = \mathbb{R}^n$; therefore, the dual variable $z_t \in E_*$ will average the one obtained on the right-hand side of the ODE

$$\dot{z}_t = -\nabla F\left(x_t\right), \; z_0 = 0. \tag{8}$$

I2. Add an inertial term to the desired mirror map from E_* to E; we write the ODE with the "mass" $\mu_t = t + \theta$, $\theta \geq 0$:

$$\mu_t \dot{x}_t + x_t = \nabla W(z_t), \quad t \geq 0, \quad x_0 = \nabla W(z_0) \in X. \tag{9}$$

Remark 1 *Trajectory* $\{x_\tau\}_{0\le\tau\le t}$*, generated by system equations* (8)*–*(9)*, is entirely contained in the set* X*. Indeed, integrating the ODE* (9)*, we obtain*

$$(t+\theta)x_t - \theta x_0 = \int_0^t \nabla W(z_\tau)\mathrm{d}\tau,$$

and, since the average $\bar{x}_t = \frac{1}{t}\int_0^t \nabla W(z_\tau)\mathrm{d}\tau \in \mathrm{X}$*, we get*

$$x_t = \frac{\theta}{t+\theta}x_0 + \frac{t}{t+\theta}\bar{x}_t \in X.$$

3 Sliding Mode Controller Synthesis

For the problem with plant (6), introduce a parameterized *sliding variable* :

$$s_t := (t+\theta)\dot{x}_t + x_t - \nabla W(z_t) \tag{10}$$

with parameters $\theta > 0$ and $\eta \in \mathbb{R}^n$, where $\{z_t\}$ is defined by ODE

$$\dot{z}_t = -\nabla F\left(x_t\right), \; z_0 = 0. \tag{11}$$

Note that equation (10) with $s_t \equiv 0$ be equivalent to (9) in I2, and the desired sliding mode is precisely with $s_t \equiv 0$.

Remark 2 *Due to assumptions A1 and A3 sliding variable* (10) *contains an integral term, and therefore its dynamic properties are similar to the behavior of a similar variable in the integral sliding mode (Fridman etal, 2014) [8].*

Let us choose control u_t as follows:

$$\left.\begin{aligned} u_t &= v_t - \frac{w_t}{t+\theta} \\ \\ w_t &= 2\dot{x}_t + \nabla^2 W(z_t)\nabla F(x_t) \\ \\ v_t &= -kR_t \mathrm{Sign}\,(s_t) \end{aligned}\right\} \tag{12}$$

$$
\left.
\begin{aligned}
&\operatorname{Sign}(z) := \left(\operatorname{sign}(z_1), \ldots, \operatorname{sign}(z_n)\right)^{\mathsf{T}} \\
&\operatorname{sign}(z_i) := \begin{cases} 1, & \text{если} \quad z_i > 0, \\ \nu_i \in [-1, 1], & \text{если} \quad z_i = 0, \\ -1, & \text{если} \quad z_i < 0, \end{cases}
\end{aligned}
\right\} \tag{13}
$$

$$
R_t := c_0 + c_1 \left\| x_t \right\|_2 + c_2 \left\| \dot{x}_t \right\|_2 . \tag{14}
$$

Analysis of the Lyapunov function. Introdce candidate

$$V(s_t) := \frac{1}{2}\left\|s_t\right\|_2^2 = \frac{1}{2}s_t^\intercal s_t. \tag{15}$$

Due to (6), (7), and (10), one has

$$\begin{aligned}\frac{d}{dt}V(s_t) &= s_t^\intercal \dot{s}_t = s_t^\intercal\left[(t+\theta)(f(x_t,\dot{x}_t,t)+u_t)+2\dot{x}_t\right.\\&\qquad\left.+\nabla^2 W(z_t)\nabla F(x_t)\right]\\&\leq (t+\theta)\left\|s_t\right\|_2\left(c_0+c_1\left\|x_t\right\|_2+c_2\left\|\dot{x}_t\right\|_2\right)-kR_t\sum_{i=1}^{n}\left|s_{i,t}\right|\\&\leq (t+\theta)\left\|s_t\right\|_2 R_t\left(1-k\right),\end{aligned} \tag{16}$$

where we used the inequality

$$\sum_{i=1}^{n} |s_{i,t}| \geq \|s_t\|_2 \,;$$

from (16) leads

$$\begin{aligned} \frac{d}{dt}V(s_t) &\leq -(t+\theta)\,\|s_t\|_2\,R_t\,(k-1) \\ &= -\sqrt{2}(t+\theta)R_t\,(k-1)\,\sqrt{V(s_t)}. \end{aligned} \tag{17}$$

4 Basic results

Desired optimization mode.

Theorem 1 *Let the assumptions A0–A3 be fulfilled, dynamic system* (6) *be closed by the controller* (12) – (14) *with the use of the sliding variable* (10) *and ODE* (11), *and let coefficient*

$$k > 1. \tag{18}$$

Then the differential inequality

$$\frac{d}{dt} V(s_t) \leq -\rho(t+\theta)\sqrt{V(s_t)}$$

with function $V(\cdot)$ *in* (15) *and coefficient*

$$\rho = \sqrt{2}c_0\,(k-1) > 0, \tag{19}$$

ensuring $s_t \equiv 0$, $\forall\, t \geq t_{reach} = \sqrt{\theta^2 + 2\sqrt{2}\rho^{-1}\|s_0\|_2} - \theta$. $\square$

Proof of Theorem 1 *The results directly follow from (17) and the inequality obtained after integration*

$$\sqrt{V(s_t)} - \sqrt{V(s_0)} \leq -\frac{\rho}{4}\left[(t+\theta)^2 - \theta^2\right], \quad t \geq 0, \tag{20}$$

while the LHS in (20) *is nonnegative. Hence it follows that* $V(s_t) = 0$ *for all* $t \geq t_{reach} = \sqrt{\theta^2 + 2\sqrt{2}\rho^{-1}\|s_0\|_2} - \theta$. *The theorem is proved.* $\blacksquare$

Corollary 1 *If $s_0 = 0$, i.e., the relation $x_0 = -\theta \dot{x}_0$ holds true due to (10), then $t_{reach} = 0$ and $s_t \equiv 0$, which relates to the sliding mode at each time $t \geq 0$.*

Rate of convergence by the loss function

Theorem 2 *Under conditions of Theorem 1 and due to relation $x_0 = -\theta \dot{x}_0$, the inequality holds*

$$F\left(x_t\right) - F^* \leq \frac{\Phi_\theta\left(x_0, x^*\right)}{t+\theta} \quad \forall\, t \geq 0\,, \tag{21}$$

where $\Phi_\theta\left(x_0, x^*\right) = \theta\left[F\left(x_0\right) - F^*\right] + W_*(x^*)$. $\square$

Proof of Theorem 2 *By Theorem 1 under condition $\eta = -\theta x_0$ the identities $s_t \equiv 0$ and $\dot{s}_t \equiv 0$ hold, therefore, due to (10) and (11) for any $t \geq 0$*

$$\left.\begin{array}{c} \dot{z}_t = -\nabla F\left(x_t\right) \\ \left(t+\theta\right)\dot{x}_t + x_t - \nabla W(z_t) = 0 \end{array}\right\} \qquad (22)$$

Hence, the trajectory $\{x_t\}$ coinsides with that of IMDM (N.2018) [5] with mass

$$\left.\begin{array}{c} \dot{z}_t = -\nabla F\left(x_t\right), \\ \mu_t \dot{x}_t + x_t = \nabla W\left(z_t\right). \end{array}\right\} \qquad (23)$$

Leading to the approach represented in (N.2018) [5] and *using (23), we get*

$$\begin{aligned}\frac{d}{dt}\left[W\left(z_t\right)-z_t^{\intercal}x^*\right]&=\dot{z}_t^{\intercal}\left(\nabla W\left(z_t\right)-x^*\right)\\&=-\nabla^{\intercal}F\left(x_t\right)\left(\mu_t\dot{x}_t+x_t-x^*\right)\\&=-\nabla^{\intercal}F\left(x_t\right)\left(x_t-x^*\right)-\mu_t\nabla^{\intercal}F\left(x_t\right)\dot{x}_t\\&\leq F^*-F\left(x_t\right)-\mu_t\frac{d}{dt}\left[F\left(x_t\right)-F^*\right]\\&=-\frac{d}{dt}\left(\left(t+\theta\right)\left[F\left(x_t\right)-F^*\right]\right).\end{aligned}$$

Now, integrating the both hand sides of the obtained inequality ovn interval $[0,t]$, *we get*

$$\left(t+\theta\right)\left[F\left(x_t\right)-F^*\right]\leq\Phi_\theta\left(x_0,x^*\right). \tag{24}$$

The theorem is proved. ■

5 Conclusion

A new approach to online optimization of an undefined dynamic system in a sliding mode using the idea of inertial mirror descent is presented. A corresponding controller is proposed, which at each time is in the desired optimization mode. The convergence and the rate of convergence with respect to the loss function are proved.

Finally, we note that the article (Poznyak etal, 2021) [10], will be published soon developing the above approach to Lagrangian systems with illustration in robotics.

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THANKS!