Network utility maximization

Optimization & Protocols

Dmitry Pasechnyuk
July 2021

Network utility maximization by updating individual transmission rates*

Dmitry Pasechnyuk[0000−0002−1208−1659]

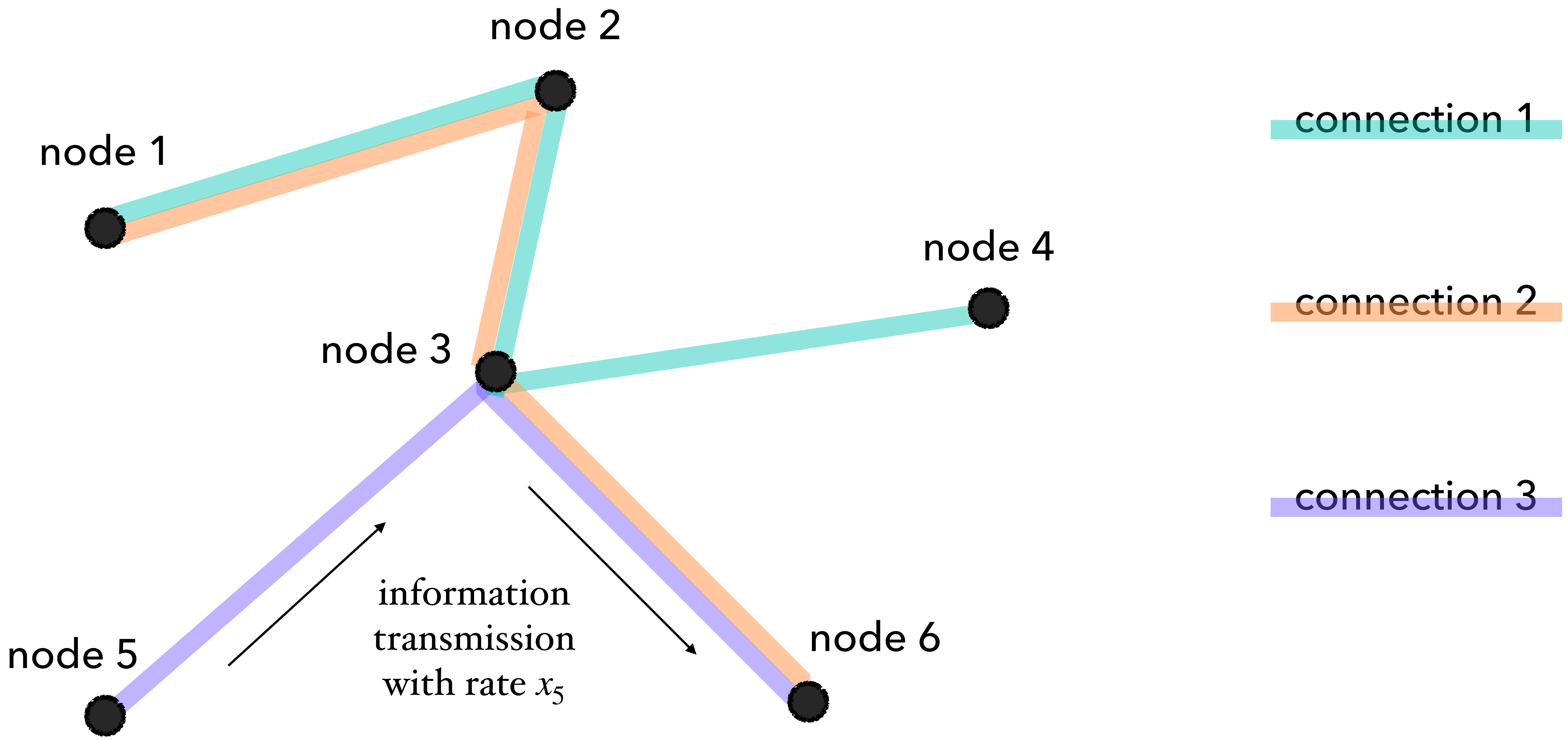
Численные методы для задачи распределения ресурсов в компьютерной сети [1]

© 2021 г. Е. А. Воронцова[1], А. В. Гасников[1,2], П. Е. Двуреченский[3,2], А. С. Иванова[4], Д. А. Пасечнюк[1]

Adaptive Mirror Descent for the Network Utility Maximization Problem*

Anastasiya Ivanova* Fedor Stonyakin**
Dmitry Pasechnyuk*** Evgeniya Vorontsova****
Alexander Gasnikov†

Problem



node 2
node 1
node 4
node 3
node 5
information
transmission
with rate x_5
node 6
connection 1
connection 2
connection 3

Problem

$$\max_{x:Cx \le b} \left\{ U(x) = \sum_{i=1}^{n} u_i(x_i) \right\}$$

n vertices, m connections

$C \in \{0,1\}^{m \times n}$ connection-defining matrix

$b \in \mathbb{R}^m$ connections throughputs

$x \in \mathbb{R}^n$ data transmission rates

$u_i(x)$ μ-strongly concave utility functions

Primal-dual approach

Economical idea

$$\min_{\lambda} \left\{ \varphi(\lambda) = \langle \lambda, b \rangle + \sum_{i=1}^{n} [u_i(x_i(\lambda)) - \langle \lambda, C_i^\top x_i(\lambda) \rangle] \right\}$$

$$x_i(\lambda) = \arg\max_{x_i} \left\{ u_i(x_i) - \langle \lambda, C_i^\top x_i \rangle \right\}$$

$$\|\nabla \varphi(\lambda_1) - \nabla \varphi(\lambda_2)\|_q \leq L \|\lambda_1 - \lambda_2\|_q, L \sim \frac{nnz(C)}{\mu}$$

Primal-dual approach

SGPM

$\boldsymbol{\lambda}^0 := \mathbf{0}$

for $t = 1, \ldots, N-1$

  Вычислить $\nabla\varphi(\boldsymbol{\lambda}^{t-1}, \xi)$

  $\boldsymbol{\lambda}^t := [\boldsymbol{\lambda}^{t-1} - \beta\left(\mathbf{b} - nC_{\xi^{t-1}}x_{\xi^{t-1}}(\boldsymbol{\lambda}^{t-1})\right)]_+$

  $\hat{\mathbf{x}}^{t+1} := \dfrac{1}{t+1}\sum_{j=0}^{t}\mathbf{x}(\boldsymbol{\lambda}^j)$

  $\hat{\boldsymbol{\lambda}}^{t+1} := \dfrac{1}{t+1}\sum_{j=0}^{t}\boldsymbol{\lambda}^j$

end for

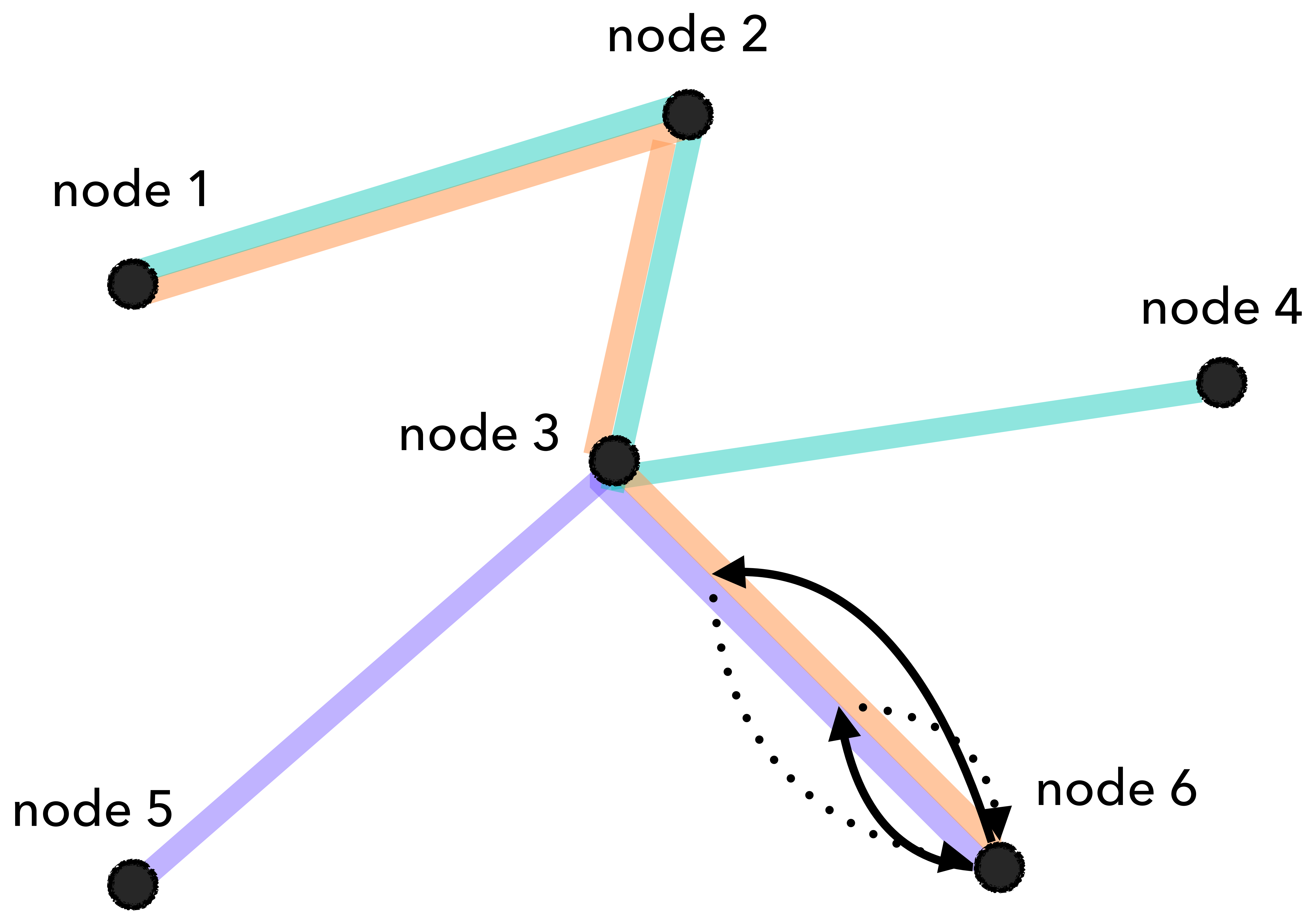
return $\hat{\boldsymbol{\lambda}}^N, \hat{\mathbf{x}}^N$

Without the smoothness assumption:

$$O\left(\frac{1}{\varepsilon^2}\log\frac{1}{\varepsilon}\right)$$

Primal-dual approach

SGPM: communication protocol



Primal-dual approach

RGEM

for $t = 1, \ldots, N$

Выбрать случайным образом k_t из множества $\{1, \ldots, n\}$ равномерно по всем значениям

$\tilde{\mathbf{y}}_k^t := \mathbf{y}_k^{t-1} + \alpha(\mathbf{y}_k^{t-1} - \mathbf{y}_k^{t-2}), \quad k = 1, \ldots, n$

$\boldsymbol{\lambda}^t := \left[\eta \boldsymbol{\lambda}^{t-1} - \frac{1}{n}\sum_{k=1}^{n} \tilde{\mathbf{y}}_k^t\right]_+ /(\delta + \eta)$

$\underline{\boldsymbol{\lambda}}_{k_t}^t := \left(\boldsymbol{\lambda}^t + \tau \underline{\boldsymbol{\lambda}}_{k_t}^{t-1}\right)/(1+\tau)$

$\underline{\boldsymbol{\lambda}}_k^t := \underline{\boldsymbol{\lambda}}_k^{t-1}, \; k \in \{1, \ldots n\} \setminus \{k_t\}$

$\mathbf{y}_{k_t}^t := \mathbf{b} - n\mathbf{C}_{k_t} x_{k_t}(\underline{\boldsymbol{\lambda}}_{k_t}^t)$

$\mathbf{y}_k^t := \mathbf{y}_k^{t-1}, \; k \in \{1, \ldots, n\} \setminus \{k_t\}$

end for

$\overline{\boldsymbol{\lambda}}^N := \left(\sum_{t=0}^{N-1} \theta_t \boldsymbol{\lambda}^t\right) / \sum_{t=1}^{N} \theta_t$

return $\overline{\boldsymbol{\lambda}}^N$

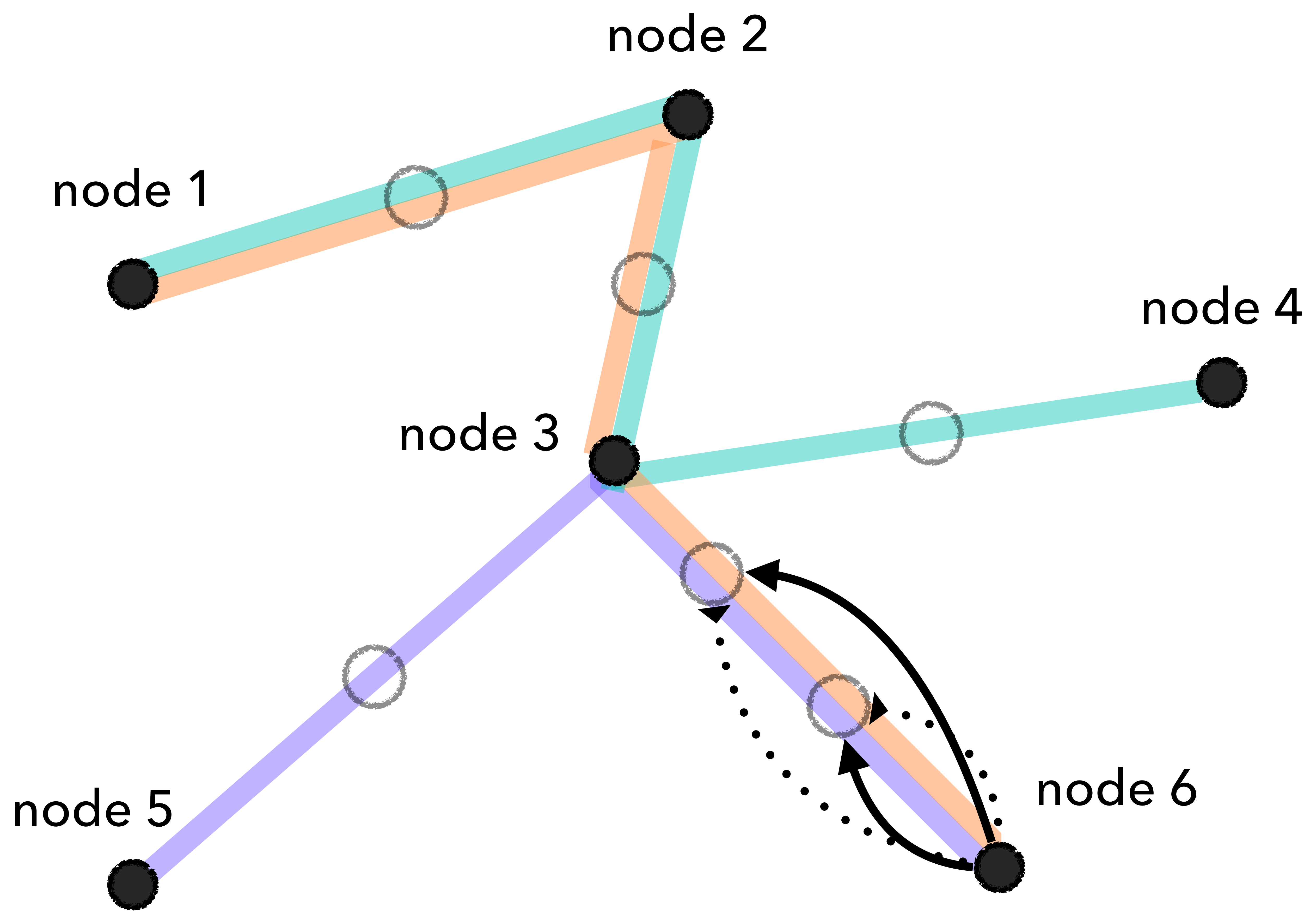
$$O\left(\frac{n}{\sqrt{\varepsilon}} \log \frac{1}{\varepsilon}\right)$$

Update only one user per iteration and all the related connections

But the method is not primal-dual, so we apply it to the regularized dual problem, it produces additional $\log(\,\cdot\,)$ factor

Primal-dual approach

RGEM: communication protocol



Primal-dual approach

FGM

Algorithm 1 Primal-dual Fast Gradient Method

Require: λ_0.

$\alpha_t = \frac{t+1}{2}$

$A_{-1} = 0,\ A_t = A_{t-1} + \alpha_t = \frac{(t+1)(t+2)}{4}$

$\tau_t = \frac{\alpha_{t+1}}{A_{t+1}} = \frac{2}{t+3}$

for $t = 0, 1, \ldots, N-1$ **do**

  Evaluate $\varphi_\mu(\lambda_t)$, $\nabla\varphi_\mu(\lambda_t)$

  $y_t = \left[\lambda_t - \frac{1}{L}\left(b - Cx(\lambda_t)\right)\right]_+$

  $z_t = \left[\lambda_0 - \frac{1}{L}\sum_{k=0}^{t} \alpha_k \left(b - Cx(\lambda_k)\right)\right]_+$

  $\lambda_{t+1} = \tau_t z_t + (1 - \tau_t) y_t$

end for

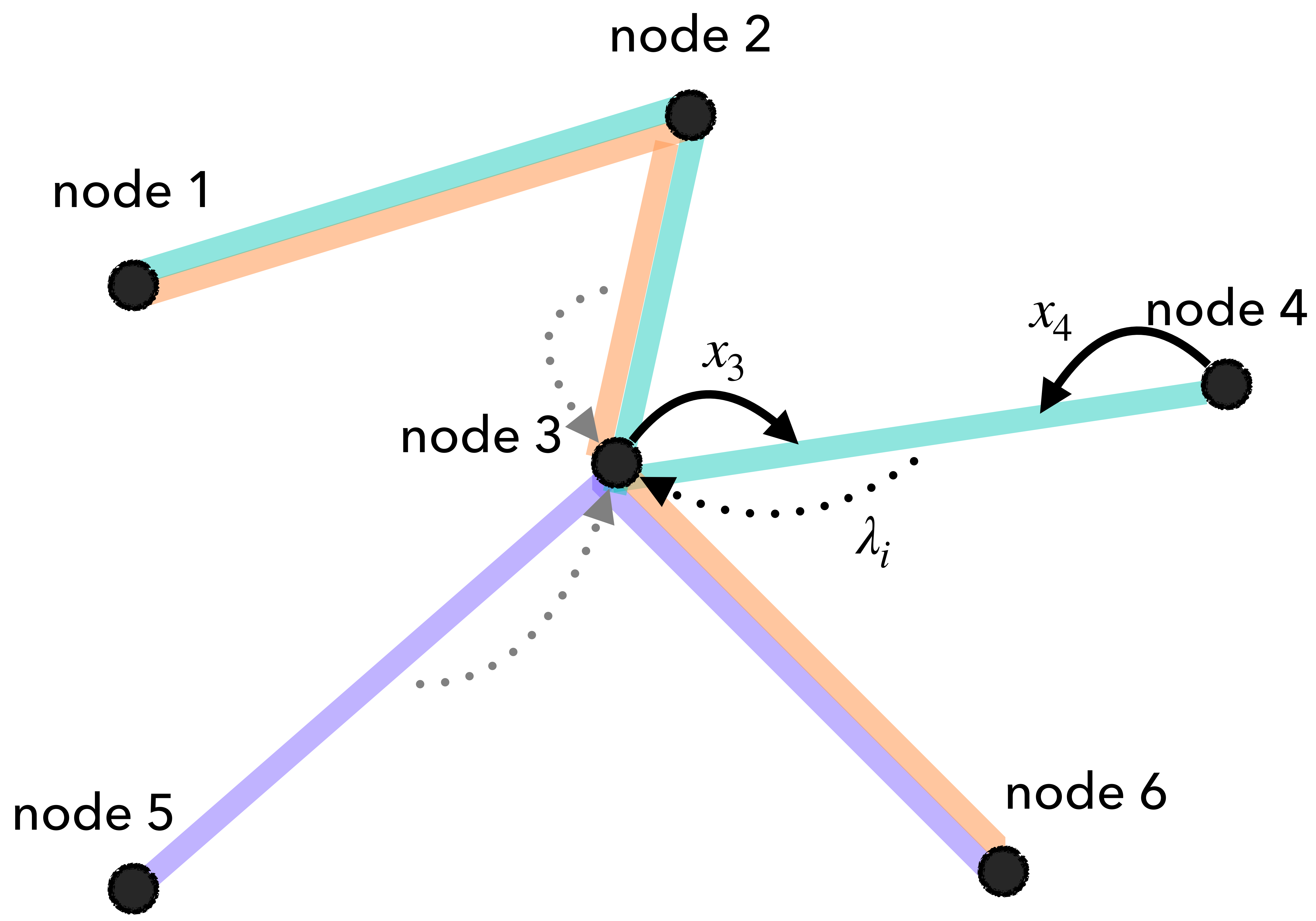
return λ_N, $\hat{x}_N = \frac{1}{A_N}\sum_{t=0}^{N} \alpha_t x(\lambda_t)$

Using Nesterov's dual smoothing technique:

$$O\left(nnz(C) \cdot \min\left\{\frac{1}{\varepsilon}, \frac{1}{\sqrt{\mu\varepsilon}}\right\}\right)$$

Primal-dual approach

FGM: communication protocol



Primal-dual approach

Ellipsoids method

Algorithm 4 Метод эллипсоидов

Вход: $u_k(x_k), k = 1, \ldots, n$ — вогнутые функции полезностей

$B_0 := 2R \cdot I_n$, I_n — единичная матрица

for $t = 0, \ldots, N-1$

  Вычислить $\nabla \varphi(\boldsymbol{\lambda}^t)$

  $\mathbf{q}_t := B_t^{\mathrm{T}} \nabla \varphi(\boldsymbol{\lambda}^t)$

  $\mathbf{p}_t := \dfrac{B_t^{\mathrm{T}} \mathbf{q}_t}{\sqrt{\mathbf{q}_t^{\mathrm{T}} B_t B_t^{\mathrm{T}} \mathbf{q}_t}}$

  $B_{t+1} := \dfrac{m}{\sqrt{m^2 - 1}} B_t + \left(\dfrac{m}{m+1} - \dfrac{m}{\sqrt{m^2 - 1}} \right) B_t \mathbf{p}_t \mathbf{p}_t^{\mathrm{T}}$

  $\boldsymbol{\lambda}^{t+1} := \boldsymbol{\lambda}^t - \dfrac{1}{m+1} B_t \mathbf{p}_t$

end for

return $\boldsymbol{\lambda}^N$

$$O\left(m^2 \log \frac{1}{\varepsilon}\right)$$

Is not communicational efficient
due to the full-vector recalculations

And we should also calculate the accuracy certificate to obtain primal solution

Switching approach

Idea

$$\max_{x: g_i(x) \leq 0} \left\{ U(x) = \sum_{i=1}^{n} u_i(x_i) \right\}$$

$$g_i(x) = C_i x - b_i$$

$$\|\nabla g_i(x)\| \leq M_g, \quad \|\nabla U(x)\| \leq M_U$$

Switching approach

AMD

repeat

  if $g_j(\mathbf{x}^N) \leqslant \varepsilon\|\nabla g_j(\mathbf{x}^N)\|_2,\ \forall j \in \overline{1,m}$ **then**

    $\mathbf{x}^{N+1} = \left[\mathbf{x}^N - \frac{\varepsilon \nabla f(\mathbf{x}^N)}{\|\nabla f(\mathbf{x}^N)\|_2^2}\right]_+$

    $// \ h_N = \frac{\varepsilon}{\|\nabla f(x^N)\|_2^2}$

    $N \to I$

  else

    $(g_{j_N}(\mathbf{x}^N) > \varepsilon\|\nabla g_{j_N}(\mathbf{x}^N)\|_2),\ j_N \in \overline{1,m})$

    $\mathbf{x}^{N+1} = \left[\mathbf{x}^N - \frac{\varepsilon \nabla g_{j_N}(\mathbf{x}^N)}{\|\nabla g_{j_N}(\mathbf{x}^N)\|_2}\right]_+$

    $// \ h_N = \frac{\varepsilon}{\|\nabla g_{j_N}(\mathbf{x}^N)\|_2}$

  end if

  $N \leftarrow N + 1$

until

$$\frac{2\Theta_0^2}{\varepsilon^2} \leqslant \sum_{k \in I} \frac{1}{\|\nabla f(x^k)\|_*^2} + |J|, \qquad (14)$$

where $|J|$ — the number of unproductive steps (we denote by $|I|$ the number of productive steps, i.e. $|I| + |J| = N$).

Ensure: $\hat{\mathbf{x}}^N = \frac{1}{\sum_{k \in I} h_k} \sum_{k \in I} h_k \mathbf{x}^k$

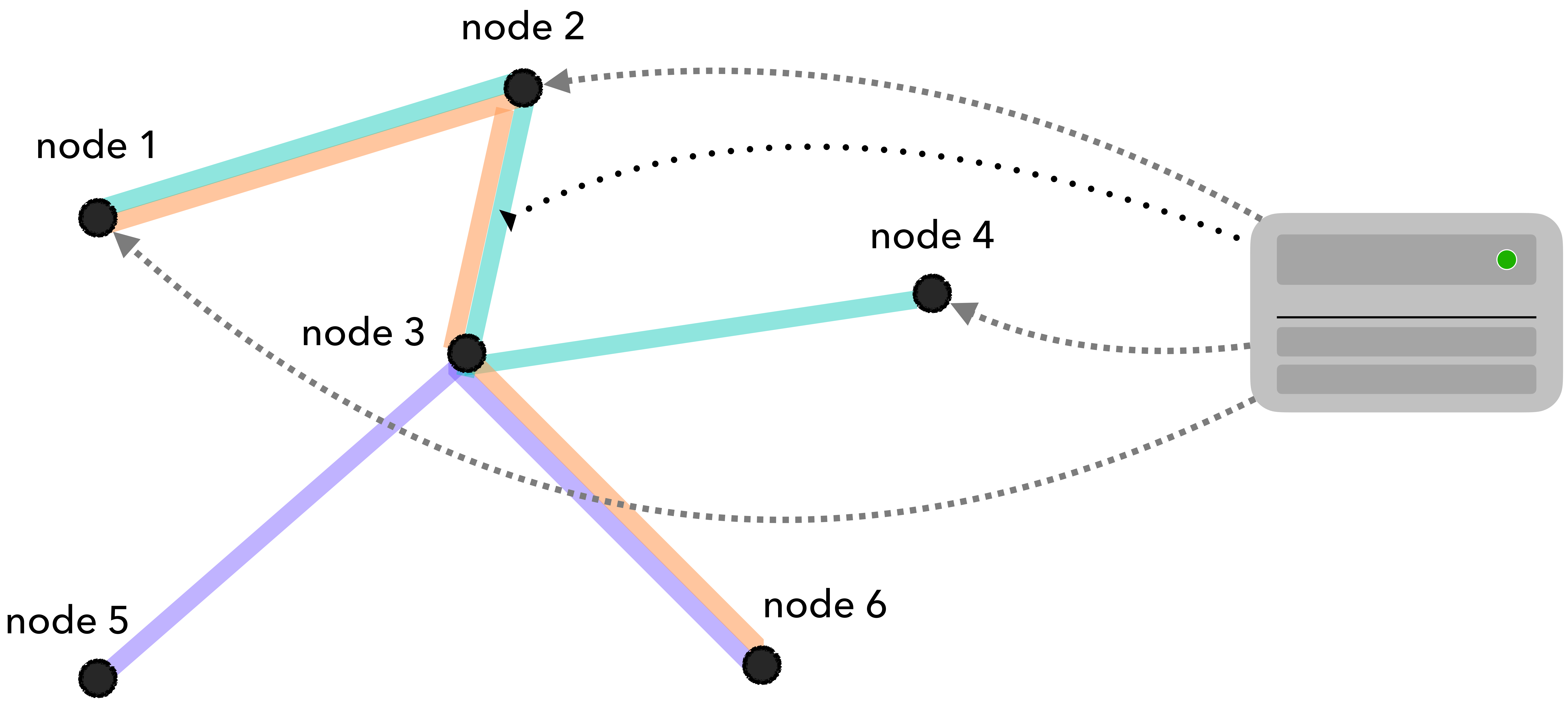
$$O\left(\max\left\{\frac{M_U^2}{\varepsilon^2}, \frac{1}{\varepsilon^4}\right\}\right)$$

Fully adaptive and do not utilize the smoothness

Requires the central manager for switching, since do not use the prices framework

Switching approach

AMD: communication protocol



Switching approach

SMD

Algorithm 2 Stochastic Mirror Descent

Require: x_0.

$I = \varnothing, J = \varnothing$
for $t = 0, 1, \ldots, N-1$ **do**
  if $Cx_t - b \leq \varepsilon$ **then**
    $i \sim \mathcal{U}\{1, ..., n\}$
    $[x_{t+1}]_i = \left[[x_t]_i - \frac{\varepsilon n}{M_U^2} \nabla u_i([x_t]_i)\right]_+$
    $I = I \cup \{t+1\}$
  else
    $j_t = \arg\max_{j=1,\ldots,m} C_j x_t - b_j$
    $i \sim \mathcal{U}\{i : C_{j_t i} = 1\}$
    $[x_{t+1}]_i = \left[[x_t]_i - \frac{\varepsilon n}{\max_{j=1,\ldots,m} \|C_j\|_{p*}^2}\right]_+$
    $J = J \cup \{t+1\}$
  end if
end for
return $\hat{x}_N = \frac{1}{|I|} \sum_{t \in I} x_t$

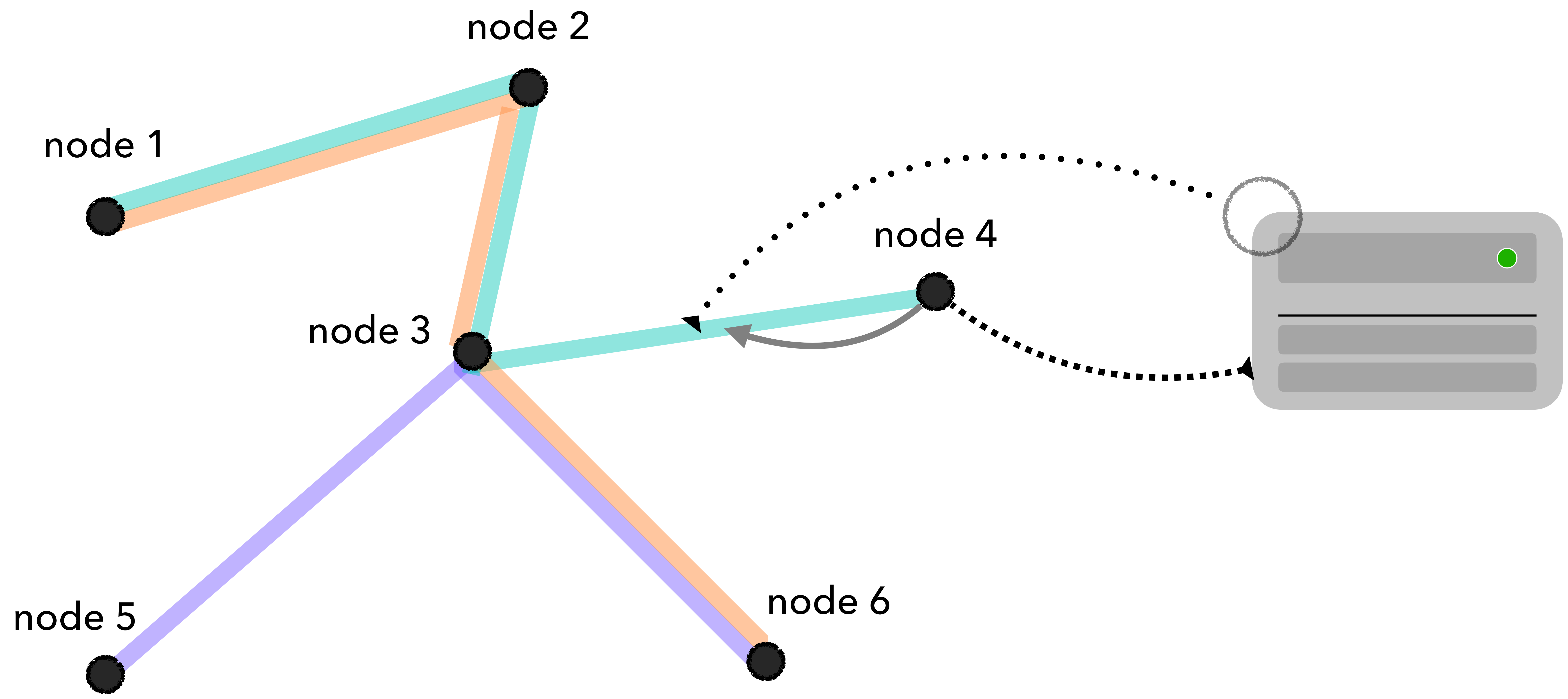
$$O\left(\frac{M_U^2 n^2}{\varepsilon^2}\right)$$

More flexible operation with a central manager due to the stochasticity

Also allowing the adaptive modifications

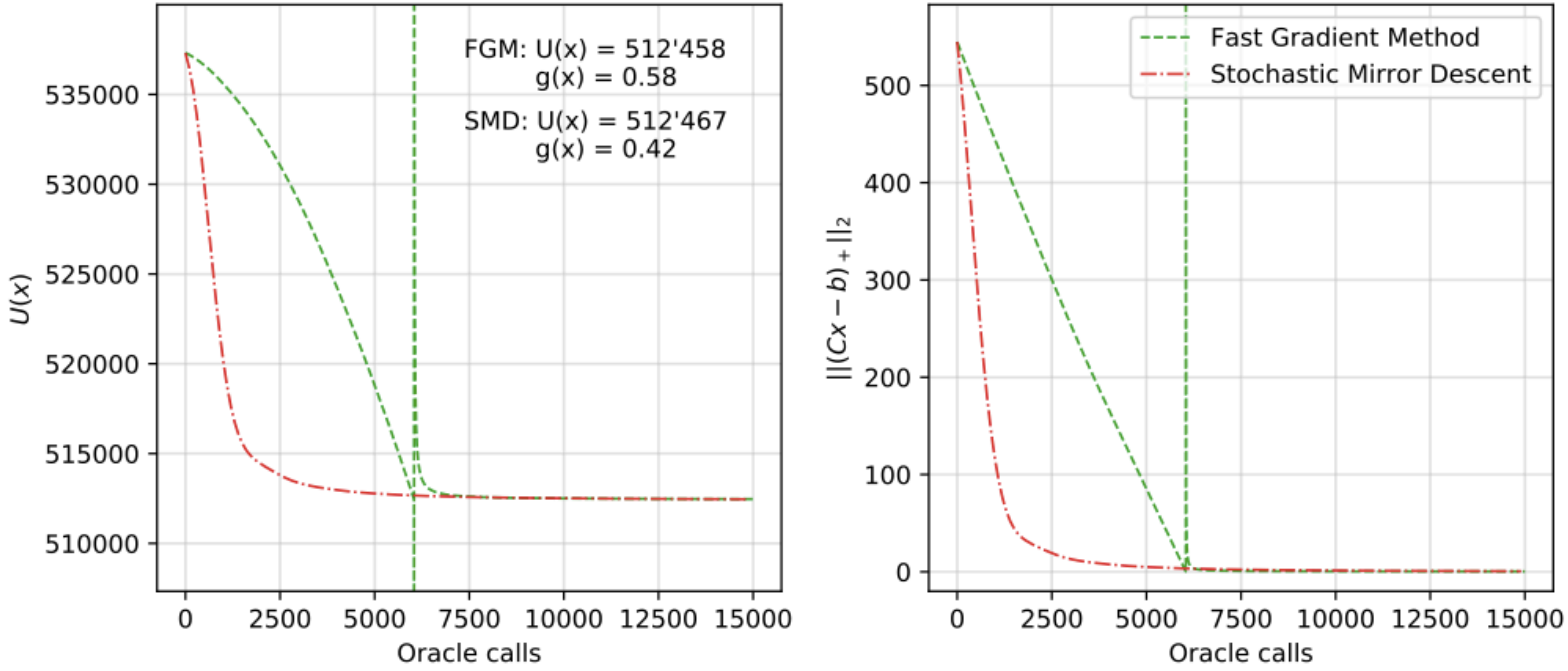
Switching approach

SMD: communication protocol



Primal-dual vs Switching

Numerical experiment



Thank you!